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AUGMENTED GROUP SYSTEMS AND SHIFTS OF FINITE TYPE

BY

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ABSTRACT

Let (G,χ,x) be a triple consisting of a finitely presented group G, epimorphism $\chi\colon G \to \mathbf{Z}$, and distinguished element $x \in G$ such that $\chi(x) = 1$. Given a finite symmetric group S_r, we construct a finite directed graph Γ that describes the set Φ_r of representations $\rho\colon \operatorname{Ker}\chi \to S_r$ as well as the mapping $\sigma_x\colon \Phi_r \to \Phi_r$ defined by $(\sigma_x\rho)(a) = \rho(x^{-1}ax)$ for all $a \in \operatorname{Ker}\chi$. The pair (Φ_r,σ_x) has the structure of a shift of finite type, a well-known type of compact 0-dimensional dynamical system. We discuss basic properties and applications of the *representation shift* (Φ_r,σ_x), including applications to knot theory.

Introduction

Assume that G is a finitely presented group and $\chi\colon G \to \mathbf{Z}$ is an epimorphism. We will denote the kernel of χ by K_χ. Efforts to understand the structure of K_χ are hampered by the fact that often this group is not finitely generated. Nevertheless, a number of applications of group theory to topology require such an investigation.

An **augmented group system** is a triple $(G,\ \chi,\ x)$ consisting of a finitely presented group G, an epimorphism $\chi\colon G \to \mathbf{Z}$, and a distinguished element $x \in G$ such that $\chi(x) = 1$. Augmented group systems were previously defined and studied in [Si1]. In this paper we show that any augmented group system $(G,\ \chi,\ x)$ determines a sequence of shifts of finite type $(\Phi_r,\ \sigma_x)$, where r is any positive integer. The elements of $(\Phi_r,\ \sigma_x)$ are the representations of K_χ

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in the symmetric group S_r. Using the tools of combinatorial group theory we give an algorithm for determining the shifts (Φ_r, σ_x). Each shift (Φ_r, σ_x) maps onto a dynamical system $(\tilde{\Phi}_r, \tilde{\sigma}_x)$, the elements of which are the subgroups $H \leq K_\chi$ having index $|K_\chi: H| \leq r$. Consequently, when (Φ_r, σ_x) is finite, K_χ contains only finitely many subgroups of index less than or equal to r. We give sufficient conditions in terms of the Bieri–Neumann–Strebel invariant [BiNeSt] for the shifts (Φ_r, σ_x) to be finite for all r. This condition depends only on the pair (G, χ). In the last section of the paper we use the entropy of the shifts (Φ_r, σ_x) to define a sequence of numerical invariants for any pair (G, χ). For the special case of a knot group G and abelianization homomorphism χ we obtain a sequence of knot invariants that are effectively computable.

1. Permutation representations

Recall that a **permutation representation** of a group K is a homomorphism $\rho: K \to S_r$, where S_r is the symmetric group operating on the set $\{1, \ldots, r\}$. We will call ρ a **representation of K in S_r**. The representation ρ is **transitive** if $\rho(K)$ operates transitively on $\{1, \ldots, r\}$. The following proposition is well known.

PROPOSITION 1.1: *Let K be any group and let r be a postive integer. The function $\pi: \rho \mapsto \{g \in K \mid \rho(g)(1) = 1\}$ maps the set of representations $\rho: K \to S_r$ onto the set of subgroups $H \leq K$ having index $|K: H| \leq r$. The preimage of any subgroup of index r contains exactly $(r-1)!$ transitive representations.*

Given any representation $\rho: K \to S_r$ one can obtain a set of generators for the subgroup $H = \pi(\rho)$ by the following familiar topological method. Let C^2 be a 2-complex with a single 0-cell v such that $\pi_1(C^2, v) \cong K$. The complex C^2 is easily constructed from a presentation (not necessarily finite) for K: oriented 1-cells in C^2 correspond to generators, oriented 2-cells correspond to relators. Use the representation ρ to build an r-sheeted covering space $p: \tilde{C}^2 \to C^2$ in the following way. The unique 0-cell v is covered by 0-cells $\tilde{v}_1, \ldots, \tilde{v}_r$. An oriented 1-cell e is covered by an oriented 1-cell that travels from $\tilde{v}_i$ to $\tilde{v}_{\rho(e)(i)}, 1 = 1, \ldots, r$; it will be helpful to label each of these 1-cells by e. Finally, each oriented 2-cell of C^2 is covered by r oriented 2-cells in an obvious manner. Let $\tilde{C}_0^2$ be the component of $\tilde{C}^2$ that contains $\tilde{v}_1$. The projection p induces a monomorphism $p_*: \pi_1(\tilde{C}_0^2, \tilde{v}_1) \to \pi_1(C^2, v)$ with image H. Projecting generators for $\pi_1(\tilde{C}_0^2, \tilde{v}_1)$ produces generators for H.

Example 1.2: Let K be the 1-relator group $\langle x, a \mid xax^{-1} = a^2 \rangle$ with representation ρ: $K \to S_2$ given by $x \mapsto (1,2)$, $a \mapsto (1)$. The 1-skeleton of C^2 and its covering space $\tilde{C}^2$ appear in Figure 1. From these we see that the elements a, xax^{-1}, x^2 generate the corresponding subgroup $H \leq K$ of index 2. Using the relations in K we obtain the more efficient set of generators $\{x^2, a\}$.

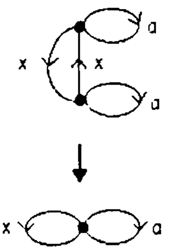

Figure 1.

In this paper we are concerned with the case $K = K_\chi$, where χ is the epimorphism of some augmented group system (G, χ, x). Given any finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ for G a well-known algorithm from combinatorial group theory, the Reidemeister–Schreier method, enables us to find a presentation (possibly infinite) for K_χ. For the convenience of the reader we review the procedure for our situation. Much of our notation comes from [Ra]. The distinguished element x corresponds to some word w in the generators x_i. We add a new symbol x and relator $x = w$ to our presentation for G. (Such an addition is called a "Tietze transformation." See [LySc].) Next we replace each generator x_i by $a_i = x_i x^{-\chi(x_i)}$. More rigorously, we introduce new symbols a_i and (defining) relators $a_i = x_i x^{-\chi(x_i)}$; the new relators are equivalent to $x_i = a_i x^{\chi(x_i)}$ which we use to rewrite $r_1, \ldots, r_m$ in terms of $x, a_1, \ldots, a_n$; finally, we eliminate the old symbols x_i and the relators $a_i = x_i x^{-\chi(x_i)}$. (The last step can be thought of as the reverse of the type of Tietze transformation with which we began.) For each $i = 1, \ldots, n$ and $j \in \mathbf{Z}$, we denote the element $x^{-j} a_i x^j$ by the symbol $a_{i,j}$. Clearly, each $a_{i,j}$ is an element of K_χ. In fact, it is not difficult to see that these elements generate K_χ. We obtain a set of relators for K_χ by rewriting each of $x^{-j} r_1 x^j, \ldots, x^{-j} r_m x^j$ as a word in the $a_{i,j}$, a rewriting that is possible because the exponent sum of x in it is zero. Notice that the rewrite of $x^{-j-t} r_k x^{j+t}$ is the just the result of adding t to the second subscripts of the rewrite of $x^{-j} r_k x^j$.

Example 1.3: Denote the group $\langle x, a \mid xax^{-1} = a^2 \rangle$ of the previous example by G. Using the Reidemeister–Schreier method one checks that the kernel K_χ of the abelianization homomorphism $\chi: G \to \mathbf{Z}$ is $\langle a_i \mid a_i = a_{i+1}^2,\ i \in \mathbf{Z}\rangle$. Consider the assignment

$$\rho(a_i) = \begin{cases} (1,2,3), & \text{if } i \text{ is even;} \\ (1,3,2), & \text{if } i \text{ is odd.} \end{cases}$$

Since $\rho(a_i) = \rho(a_{i+1}^2)$ for all $i \in \mathbf{Z}$, the function ρ induces a representation of K_χ in S_3. The 1-skeletons of C^2 and $\tilde{C}^2$ are indicated in Figure 2. The corresponding subgroup $H \leq K_\chi$ of index 3 is generated by

$$a_{2i}a_{2j+1}, a_{2i+1}a_{2j}, a_{2i}a_{2j}a_{2k}, a_{2i+1}a_{2j+1}a_{2k+1}$$

where i, j and k range over all integers. The relations in K enable us to obtain the relatively smaller set of generators $\{a_i^3 \mid i \in \mathbf{Z}\}$. In fact, K_χ is isomorphic to the group $\mathbf{Z}[1/2]$ of the dyadic rationals via the isomorphism $a_i \mapsto 1/2^i$. Under this isomorphism the subgroup H is simply $3\mathbf{Z}[1/2]$.

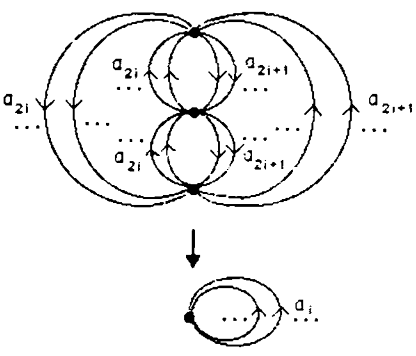

Figure 2.

In general, if ρ is any representation of K in S_r, then the index of $\pi(\rho)$ is the cardinality of the maximal subset of $\{1, \ldots, r\}$ containing 1 on which $\rho(K)$ operates transitively. Given two representations, ρ_1 and ρ_2, it is a simple task to determine whether the subgroups $H_1 = \pi(\rho_1)$ and $H_2 = \pi(\rho_2)$ are equal. First, we check whether H_1 and H_2 have the same index j in K. If that is the case, then we can assume without any loss of generality that ρ_1 and ρ_2 are transitive representations of K in S_j. Now H_1 and H_2 are equal if and only if $\rho_2 = \tau \circ \rho_1$ for some inner automorphism τ of S_r, conjugation by a permutation that fixes 1.

2. The dynamical systems of $(G,\ \chi,\ x)$

By a **dynamical system** we will mean a pair $(X,\ \sigma)$ consisting of a topological space X and a homeomorphism $\sigma: X \to X$. A mapping $f: (X,\ \sigma) \to (X',\ \sigma')$ of dynamical systems is a continuous function $f\colon X \to X'$ for which $f \circ \sigma = \sigma' \circ f$. The dynamical systems $(X,\ \sigma)$ and $(X',\ \sigma')$ are **conjugate** if also there exists a mapping $g\colon (X',\ \sigma') \to (X,\ \sigma)$ such that $g \circ f$ and $f \circ g$ are the identity functions.

Definition 2.1: Let r be a positive integer. The **representation shift** associated to $(G,\ \chi,\ x)$ is the dynamical system $(\Phi_r,\ \sigma_x)$ consisting of the space Φ_r of representations $\rho\colon K_\chi \to S_r$ and mapping σ_x described by $\sigma_x(\rho)(g) = \rho(x^{-1}gx)$, for all $g \in K_\chi$. The topology on Φ_r is determined by the basis sets $\mathcal{N}_{g_1,\ \dots,\ g_n}(\rho) = \{\rho' \mid \rho'(g_i) = \rho(g_i),\ i = 1,\ \dots,\ n\}$, for all $\rho \in \Phi_r,\ g_1,\ \dots,\ g_n \in K_\chi$.

We leave it to the reader to check that σ_x is indeed a homeomorphism. The term "representation shift" will be justified by Theorem 3.1 in the next section.

A mapping $h\colon (G,\ \chi,\ x) \to (G',\ \chi',\ x')$ of augmented group systems is a homomorphism $h\colon G \to G'$ such that $h(x) = x'$ and $\chi = \chi' \circ h$. It is easy to check that such a homomorphism induces a mapping $h^*\colon (\Phi',\ \sigma'_x) \to (\Phi,\ \sigma_x)$ between the associated representation shifts. The mapping h^* is described by $\rho' \mapsto \rho' \circ h$.

Definition 2.2: Let r be a postive integer. The **subgroup system** associated to $(G,\ \chi,\ x)$ is the dynamical system $(\tilde{\Phi}_r,\ \tilde{\sigma}_r)$ consisting of the space $\tilde{\Phi}_r$ of subgroups $H \le K_\chi$ with $|K_\chi\colon H| \le r$ and mapping $\tilde{\sigma}_x\colon H \mapsto x^{-1}Hx$. The topology on $\tilde{\Phi}_r$ is determined by the basis sets

$$\mathcal{N}_{g_1,\ \dots,\ g_n}(H) = \{H' \mid H \cap \mathrm{gp}(g_1,\ \dots,\ g_n) = H' \cap \mathrm{gp}(g_1,\ \dots,\ g_n)\},$$

for all $H \in \tilde{\Phi}_r,\ g_1,\ \dots,\ g_n \in K_\chi$. (Here $\mathrm{gp}(g_1,\ \dots,\ g_n)$ denotes the subgroup of K_χ generated by $g_1,\ \dots,\ g_n$.)

Again we leave to the reader the task of checking that the mapping is a homeomorphism. The following proposition follows easily from Proposition 1.1.

PROPOSITION 2.3: *Let r be a positive integer. The function $\pi\colon \rho \mapsto H = \{g \in K_\chi \mid \rho(g)(1) = 1\}$ induces a mapping from $(\Phi_r,\ \sigma_r)$ onto $(\tilde{\Phi}_r,\ \tilde{\sigma}_r)$.*

The following is a consequence of Propositions 1.1 and 2.3.

COROLLARY 2.4: *Let $(G,\ \chi,\ x)$ be an augmented group system and let r be a positive integer. Then the associated representation shift $(\Phi_r,\ \sigma_x)$ is finite (resp., countably infinite, uncountable) if and only if the subgroup system $(\tilde{\Phi}_r,\ \tilde{\sigma}_x)$ is finite (resp., countably infinite, uncountable).*

3. Shifts of finite type

Example 1.3 suggests that if χ is a homomorphism of a finitely presented group G onto $\mathbf{Z}$, then the kernel K_χ can be very complicated. However, any presentation of K_χ that one obtains from the Reidemeister–Schreier method has structure that suggests a dynamical system. We pursue this line of thought, and show that every representation shift is conjugate to a shift of finite type.

Briefly we review some definitions and facts from symbolic dynamics. For more details, see [LiMa]. Let $\mathcal{A}$ be any finite set. We call $\mathcal{A}$ an **alphabet** and its elements **letters.** We give $\mathcal{A}$ the discrete topology and $\mathcal{A}^{\mathbf{Z}}$ the product topology. The **shift map** σ on $\mathcal{A}^{\mathbf{Z}}$ is the function that takes any $\rho = (\rho_j),\ \rho_j \in \mathcal{A}$, to $\rho' = (\rho'_j)$, where $\rho'_j = \rho_{j+1}$. Then $(\mathcal{A}^{\mathbf{Z}},\ \sigma)$ is a dynamical system; we refer to this system, or to the set $\mathcal{A}^{\mathbf{Z}}$ itself, as the **full $\mathcal{A}$-shift.** In particular, $\{0,\ 1,\ \ldots,\ r-1\}^{\mathbf{Z}}$ is called the **full r-shift.**

If X is a closed subset of $\mathcal{A}^{\mathbf{Z}}$ with $\sigma(X) = X$, then $(X,\ \sigma)$ is a dynamical system that we call a **subshift** of $\mathcal{A}^{\mathbf{Z}}$, or simply a **shift.** Again, we may call the set X a shift, with σ understood.

By a **block (over $\mathcal{A}$)** we mean any finite sequence w of letters. If the length of w is N, then w is said to be an N-block. The empty block ϵ has length zero. A block w **occurs** in $\rho \in \mathcal{A}^{\mathbf{Z}}$ if w appears as some subsequence of consecutive letters in ρ. Let $\mathcal{B}$ be a collection of N-blocks for some N. The set X of all $\rho \in \mathcal{A}^{\mathbf{Z}}$ such that every N-block occurring in ρ is in $\mathcal{B}$ is a subshift. A shift of this kind is called a **shift of finite type**, and $\mathcal{B}$ is its set of **allowed N-blocks.**

When $N = 2$ we may represent X by a directed graph Γ: the vertex set is $\mathcal{A}$, and there is an edge from ρ_0 to ρ'_0 if $\rho_0\rho'_0$ is an allowed 2-block. The points of X are in one-to-one correspondence with the bi-infinite walks in Γ. Conversely, if $\Gamma = (V,\ E)$ is a directed graph with no parallel edges, then we obtain a shift of finite type $X \subset V^{\mathbf{Z}}$ in the obvious manner; X is called the **vertex shift** with graph Γ. We also obtain a shift of finite type $\hat{X} \subset E^{\mathbf{Z}}$: elements of $\hat{X}$ are the bi-infinite sequences of edges that form paths in Γ, and ee' is an allowed 2-block if the terminal vertex of e is the initial vertex of e'. The shift $\hat{X}$ is called the **edge shift** with graph Γ. It is easy to see that X and $\hat{X}$ are conjugate shifts under the map that takes the vertex sequence of a path to the corresponding edge sequence. We remark that we can define the edge shift as above even if Γ has parallel edges. The vertex shift and edge shift are unaffected if we "prune" Γ, removing any vertex v or edge e that does not lie on any bi-infinite path in Γ.

As an example of the above, let $\mathcal{A} = \{0,\ 1,\ 2\}$ and $\mathcal{B} = \{00,\ 01,\ 10,\ 02,\ 20\}$. In Figure 3 we represent the associated shift X, first as the vertex shift of a graph Γ and then as the edge shift of another graph Γ'.

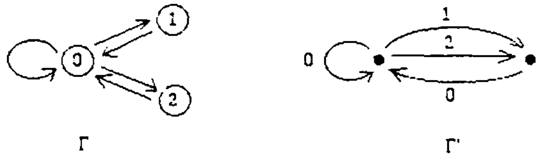

Figure 3.

THEOREM 3.1: *Assume that $(G,\ \chi,\ x)$ is an augmented group system. For any positive integer r, the associated representation shift $(\Phi_r,\ \sigma_x)$ is conjugate to a shift of finite type.*

Proof: Recall from the discussion preceding Example 1.3 that G has a presentation of the form $\langle x, a_1, \ldots, a_n \mid r_1, \ldots, r_m\rangle$ such that $\chi(x) = 1$ and $\chi(a_1) = \cdots = \chi(a_n) = 0$. Also, K_χ has a presentation

$$(3.2) \qquad \langle a_{i,j} \mid R_j, 1 \le i \le n, j \in \mathbf{Z}\rangle,$$

where the symbols $a_{i,j}$ denote generators $x^{-j}a_ix^j$, and R_j is $\{x^{-j}r_1x^j, \ldots, x^{-j}r_mx^j\}$ written as words in the generators. Recall that R_{q+t} is obtained from R_q by adding t to the second subscript of every symbol in R_q. Assume that the words in R_0 (and hence in each R_q) are reduced and cyclically reduced; i.e., no generator appears next to its inverse, and no word in R_0 ends with the inverse of the generator with which it begins. Replacing the original relators r_j by suitable conjugates $x^{-t_j}r_jx^{t_j}$, we can assume that if R_0 contains $a_{i,j}$ for some j, then R_0 contains no $a_{i,j}$ with $j < 0$. Then replacing the original generators by suitable conjugates, we can assume that if R_0 contains $a_{i,j}$ for some j, then R_0 contains $a_{i,0}$. (See [Ra] for details.) If $a_{i,0}$ occurs in R_0, then let M_i be the largest value of j such that $a_{i,j}$ occurs. If $a_{i,0}$ doesn't occur in R_0, then let M_i be zero.

From the presentation (3.2) of K_χ we obtain a presentation of some group

$$H_0 = \langle a_{1,0}, a_{1,1}, \ldots, a_{1,M_1}, a_{2,0}, \ldots, a_{n,M_n} \mid R_0\rangle.$$

Since in K_χ the generators of H_0 might satisfy relators other than those that are consequences of R_0, the group H_0 is in general not a subgroup of K_χ. Nevertheless, H_0 is valuable for studying the permutation representations of K_χ.

Abbreviate the set of generators $\{a_{1,0},\ a_{1,1},\ \ldots,\ a_{1,M_1},\ a_{2,0},\ \ldots,\ a_{n,M_n}\}$ by the symbol A_0, and let $A_t = \{a_{1,t},\ a_{1,1+t},\ \ldots,\ a_{1,M_1+t},\ a_{2,t},\ \ldots,\ a_{n,M_n+t}\}$. Combining the presentations $\langle A_t |\ R_t\rangle$ as t ranges over $\mathbf{Z}$ reproduces the presentation (3.2) of K_χ.

Deleting $a_{1,M_1},\ \ldots,\ a_{n,M_n}$ from A_0 produces a subset that we will denote by $A_{0,1}$. Similarly $A_{0,2}$ is the result of deleting $a_{1,0},\ \ldots, a_{n,0}$ from A_0.

Let $\mathcal{A}$ denote the set of all representations of H_0 in S_r. Such representations are precisely those functions ρ_0: $A_0 \to S_r$ such that the m equations $\rho_0(r_i) = \mathrm{id}$, $r_i \in R_0$, hold in S_r. In particular, $\mathcal{A}$ is a finite and computable set. Construct a directed graph Γ with vertex set $\mathcal{A}$. Draw a directed edge from vertex ρ_0 to vertex ρ_0' if and only if $\rho_0(a_{i,j+1}) = \rho_0'(a_{i,j})$ for each $a_{i,j} \in A_{0,1}$. The graph Γ determines a shift of finite type X with alphabet $\mathcal{A}$.

Any element $\rho = (\rho_j)$ of X determines a well-defined function $\bigcup_{t\in\mathbf{Z}} A_t \to S_r$ by $a_{i,j} \mapsto \rho_t(a_{i,j-t})$ if $a_{i,j} \in A_t$. This function maps each relator $x^{-t}r_jx^t$ in R_t to the element $\rho_t(r_j)$ which is the identity (since ρ_t is a homomorphism), and hence it induces a homomorphism from $K = \langle A_t \mid R_t,\ t \in \mathbf{Z}\rangle$ to S_r. It is easy to check that this determines a continuous shift-commuting function f from X to $(\Phi_r,\ \sigma_x)$.

Conversely, any representation ρ: $K \to S_r$ determines a function $A_0 \to S_r$ for each t by $a_{i,j} \mapsto \rho(a_{i,j+t})$. The function maps each relator r_j in R_0 to $\rho(x^{-t}r_jx^t)$, the identity element of S_r, and hence it induces a homomorphism ρ_t from H_0 to S_r. Clearly, $\rho = (\rho_t)$ is an element of the shift X, and we obtain a continuous shift-commuting function g: $(\Phi_r,\ \sigma_x) \to X$. Since f and g are inverses, the shifts $(\Phi_r,\ \sigma_x)$ and X are conjugate. ∎

In view of Theorem 3.1 we will regard elements of the shift X as representations ρ: $K_\chi \to S_r$ without explicit mention of the correspondence. Note that under the correspondence the allowable N-blocks of X are the representations of the group $H_{[0,N-1]} = \langle A_0,\ \ldots,\ A_{N-1} \mid R_0,\ \ldots,\ R_{N-1}\rangle$ in S_r.

Any presentation with the form (3.2) is called a **finite Z-dynamical presentation** (présentation Z-dynamique finie) in [HaKe]. The results of this paper can be stated in terms of such presentations rather than augmented group systems.

Example 3.3: Let $G = \langle x,\ a \mid xax^{-1} = a^2\rangle$ and let χ be the abelianization homomorphism. In Example 1.3 we displayed a particular representation of K_χ in S_3. Now we will find all such representations. Recall that $K_\chi =$

$\langle a_i \mid a_i = a_{i+1}^2,\ i \in \mathbf{Z}\rangle$. Here $M_1 = 1$, and $H_0 = \langle A_0 \mid R_0\rangle$, where $A_0 = \{a_0, a_1\}$ and R_0 consists of the single relator $a_0 = a_1^2$. (We have indexed the letter a with only its second subscript since its first subscript is always equal to 1.) The symmetric group S_3 is generated by $\sigma = (1,2,3)$ and $\tau = (1,2)$. There are exactly 6 representations of H_0 in S_3, corresponding to the 6 values, $(\mathrm{id},\mathrm{id}), (\sigma,\ \sigma^2),\ (\sigma^2,\ \sigma),\ (\mathrm{id},\ \tau),\ (\mathrm{id},\ \sigma\tau),\ (\mathrm{id},\ \sigma^2\tau)$, for $(a_0,\ a_1)$ that respect the relator of H_0. The pairs become vertices of the directed graph Γ, with a directed edge from vertex v to vertex v' if and only if the second coordinate of v is equal to the first coordinate of v'. The graph Γ appears in Figure 4.

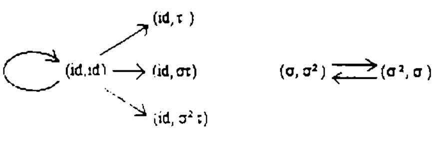

Figure 4.

From Γ we see that the shift $(\Phi_3,\ \sigma_x)$ is finite, consisting of only 3 elements. One element is a representation of K_χ in S_3 corresponding to the constant sequence $(\ldots,\ \mathrm{id},\ \mathrm{id},\ \ldots) \in X$. The subgroup of K_χ corresponding to this representation is K_χ itself. The other two elements are transitive representations of K_χ in S_3 that form an orbit of period 2 in (Φ_3, σ_x); i.e., they are sent to each other by σ_x. These two representations extend to G (see Proposition 3.5). Since one of the representations is the other composed with an inner automorphism of S_3, conjugation by $(2,3)$, both determine the same subgroup of index 3, namely $3\mathbf{Z}[1/2]$. The kernel K_χ has no subgroup of index 2. It will follow later from a more general result (Proposition 5.1) that K_χ has a subgroup of index r if and only if r is odd, and in that case such a subgroup is unique, isomorphic to $r\mathbf{Z}[1/2]$.

Although it is convenient for many arguments to represent X as the vertex shift of a directed graph Γ, we can also represent X as the edge shift of another graph $\hat{\Gamma}$ that usually has fewer vertices than Γ and therefore is simpler to compute. The vertices of $\hat{\Gamma}$ correspond to functions $\rho_0\colon A_{0,1} \to S_r$. (When $A_{0,1}$ is empty, $\hat{\Gamma}$ has a single vertex corresponding to the unique function $\rho_0\colon \emptyset \to S_r$.) If ρ_0 is a representation of H_0 in S_r, we draw a directed edge labeled by ρ_0 from the vertex labeled by $\rho_0|_{A_{0,1}}$ to the vertex labeled by $\rho_0'|_{A_{0,1}}$, where $\rho_0'(a_{i,j}) = \rho_0(a_{i,j+1})$ for all $a_{i,j} \in A_{0,1}$. Notice that the edges of $\hat{\Gamma}$ correspond to the vertices of Γ. Unlike Γ, the directed graph $\hat{\Gamma}$ might have parallel directed edges joining a pair of vertices. (This occurs when some M_i is zero so that no $a_{i,j}$ appears in $A_{0,1}$ or $A_{0,2}$.) We illustrate with an example that shows that the set of representations of K_χ in S_r can be considerably larger than the set of representations of G in S_r, in contrast to the previous example.

Example 3.4: Let $G = \langle x,\ a_1,\ a_2 \mid x^{-1}a_1^2x = a_1^{-1}a_2^2a_1\rangle$ and consider the homomorphism χ: $G \to \mathbf{Z}$ described by $x \mapsto 1$, $a_1 \mapsto 0$, and $a_2 \mapsto 0$. It is clear from the form of the relation $x^{-1}a_1^2x = a_1^{-1}a_2^2a_1$ that any function from the set of generators $\{x,\ a_1,\ a_2\}$ into the symmetric group $S_2 = \{\mathrm{id},\ \sigma\}$ induces a homomorphism. Hence there are exactly 8 representations of G in S_2. In order to find all representations of K_χ, we first apply the Reidemeister–Schreier method to obtain $K_\chi = \langle a_{1,j}, a_{2,j} | \ a_{1,j}^{-1}a_{2,j}^2a_{1,j} = a_{1,j+1}^2\rangle$. Then $M_1 = 1$, $M_2 = 0$, and $H_0 = \langle A_0 |\ R_0\rangle$, where $A_0 = \{a_{1,0}, a_{1,1}, a_{2,0}\}$ and R_0 consists of the single relator $a_{1,0}^{-1}a_{2,0}^2a_{1,0} = a_{1,1}^2$. The directed graph $\hat{\Gamma}$ has 2 vertices, corresponding to the functions from $A_{0,1} = \{a_{1,0}\}$ to S_2. Again from the form of the relator there are 8 representations ρ_0: $H_0 \to S_2$, corresponding to ordered triples of elements in S_2, the values of $(a_{1,0},\ a_{1,1},\ a_{2,0})$. The eight triples become edge labels. The directed graph $\hat{\Gamma}$ appears in Figure 5. From it we see that $(\Phi_2,\ \sigma_x)$ is uncountable. Hence K_χ has uncountably many representations in S_2 and uncountably many subgroups of index 2 by Theorem 3.1 and Corollary 2.4.

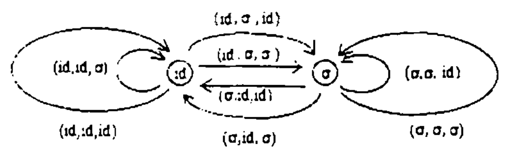

Figure 5.

When does a representation of ρ: $K_\chi \to S_r$ extend to a representation of G in S_r? Proposition 3.5 gives a complete answer. Corollary 3.6 provides a necessary condition in terms of $(\Phi_r,\ \sigma_x)$.

PROPOSITION 3.5: *A representation ρ: $K_\chi \to S_r$ extends to a representation of G in S_r if and only if there exists an element $\tau \in S_r$ such that $\tau^{-1}\rho(a_{i,j})\tau = \rho(a_{i,j+1}), 1 \le i \le n,\ j \in \mathbf{Z}$.*

Proof: Suppose that ρ: $K_\chi \to S_r$ extends to a representation ρ: $G \to S_r$. Let $\tau = \rho(x)$. Applying ρ to each side of the relation $x^{-1}a_{i,j}x = a_{i,j+1}$, we see that $\tau^{-1}\rho(a_{i,j})\tau = \rho(a_{i,j+1})$.

Conversely, if ρ is a representation of K_χ in S_r and there exists an element

$\tau \in S_r$ such that $\tau^{-1}\rho(a_{i,j})\tau = \rho(a_{i,j+1}), 1 \le i \le n, j \in \mathbf{Z}$, then we can extend ρ to all of G by defining $\rho(x)$ to be τ. ∎

COROLLARY 3.6: *If a representation $\rho: K_\chi \to S_r$ extends to a representation of G in S_r, then ρ is a periodic point of $(\Phi_r,\ \sigma_x)$; i.e., $\sigma_x^d(\rho) = \rho$ for some positive integer d.*

Proof: Let d be the order of τ in S_r. Since $\rho(a_{i,j}) = \tau^{-d}\rho(a_{i,j})\tau^d = \rho(a_{i,j+d})$, it follows that $\sigma^d(\rho) = \rho$ in the shift (Φ_r, σ_x). ∎

We remark that periodic points of $(\Phi_r,\ \sigma_x)$ need not extend to representations of G in S_r. For example, one easily checks in Example 3.4 that the representation corresponding to $\rho = (\rho_t)$, where

$$\rho_t = \begin{cases} (\mathrm{id},\ \sigma,\ \mathrm{id}), & \text{if } t \text{ is even,} \\ (\sigma,\ \mathrm{id},\ \mathrm{id}), & \text{if } t \text{ is odd,} \end{cases}$$

does not extend to a representation of G in any symmetric group S_r. However, it is possible to characterize the periodic points of a representation shift. If $(G,\ \chi,\ x)$ is an augmented group system, then for any positive integer d we let $K_{\chi,d}$ denote the kernel of the composition $G \to \mathbf{Z} \to \mathbf{Z}/(d)$, where the second mapping is the natural quotient projection.

PROPOSITION 3.7: *A representation $\rho: K_\chi \to S_r$ is a periodic point of $(\Phi_r,\ \sigma_x)$ if and only if ρ extends to a representation of $K_{\chi,d}$ for some $d > 0$.*

Proof: Assume that $\langle a_{i,j} | \ R_j, 1 \le i \le n, j \in \mathbf{Z}\rangle$ is a presentation for K_χ. It is not difficult to see that $K_{\chi,d}$ has presentation $\langle y, a_{i,j} |\ R_j, y^{-1}a_{i,j}y = a_{i,j+d}, 1 \le i \le n, j \in \mathbf{Z}\rangle$ in which y represents the element $x^d \in G$. If $\sigma_x^d(\rho) = \rho$, then we can extend ρ to $K_{\chi,d}$ by defining $\rho(y) = \mathrm{id}$. Conversely, if $\rho: K_\chi \to S_r$ extends to a representation of $K_{\chi,d}$ in S_r then, denoting the extension by ρ, we must have $\rho(y)^{-1}\rho(a_{i,j})\rho(y) = \rho(a_{i,j+d})$ for all $a_{i,j}$. Let q be the order of $\rho(y)$ in S_r. Then $\sigma_x^{dq}\rho(a_{i,j}) = \rho(a_{i,j+dq}) = \rho(y)^{-dq}\rho(a_{i,j})\rho(y)^{dq} = \rho(a_{i,j})$. Hence ρ is periodic. ∎

4. The Bieri–Neumann–Strebel invariant

Again let G be a finitely presented group and let χ be an epimorphism with kernel K_χ. Given a finite presentation of G, we obtained in Section 3 a presentation $\langle A_0 |\ R_0\rangle$ of a certain group H_0. Although, as we remarked, H_0 is not necessarily a subgroup of K_χ, some quotient group H_0^* is. In [Ra] Rapaport describes how one recovers G using H_0^*: Let $H_{0,1}^*$ be the subgroup of H_0^* generated by $A_{0,1}$, and let $H_{0,2}^*$ be the subgroup generated by $A_{0,2}$ (see the proof of Theorem 3.1 for the definitions of $A_{0,1}$ and $A_{0,2}$). The mapping $a_{i,j} \mapsto a_{i,j+1}$ induces an isomorphism $\phi\colon H_{0,1}^* \xrightarrow{\sim} H_{0,2}^*$. The group G can be described as $\langle x, H_0^* | \; x^{-1}hx = \phi(h), h \in H_{0,1}^*\rangle$ (where, abusing notation in the usual way, we write H_0^* instead of specific generators and relators for that group). Connoisseurs will recognize G as an HNN extension of H_0^*. We recall that a group G is an *HNN* **extension** of a group B if there exist subgroups S and T of B and an isomorphism $\phi\colon S \xrightarrow{\sim} T$ such that $G = \langle x, B \mid x^{-1}sx = \phi(s), \;\; s \in S\rangle$, where x is a generator not contained in B. In this case, B is called the **base** of the HNN extension, while S and T are the **associated subgroups.** (See [LySc] for additional details.) If it is the case that S coincides with the base B, then the HNN extension is said to be **ascending.**

If G is any finitely presented group and $\chi\colon G \to \mathbf{Z}$ is an epimorphism, then G can be described as an HNN extension with finitely generated base B contained in K_χ (see [BiSt].) In [BiNeSt] Bieri, Neumann and Strebel show that if one such HNN extension describing G is ascending, then all are. Indeed, they show that this is the case if and only if the class $[\chi]$ in $(\mathrm{Hom}(G, \mathbf{R}) - \{0\})/\mathbf{R}_+$, where $\mathbf{R}_+$ acts by multiplication, lies in a certain subset Σ. The subset Σ has a geometric interpretation in terms of the Cayley complex of G, and it has been generalized by Renz [Re] to a chain of "higher geometric invariants."

THEOREM 4.1: *Assume that (G, χ, x) is an augmented group system. If $[\chi] \in \Sigma$, then for any r the associated shift (Φ_r, σ_x) is finite. Consequently, K_χ contains only finitely many subgroups H with index $|K_\chi\colon H| \le r$ for any $r < \infty$.*

Notice that the hypothesis of Theorem 4.1 makes no mention of the distinguished element x. In fact, Theorem 4.1 is a result about pairs (G, χ) such that G is a finitely presented group and $\chi\colon G \to \mathbf{Z}$ is an epimorphism. We will call such a pair a **group system.**

Before proving Theorem 4.1, we present an application. Recall that a group G is **residually finite** if the intersection of all finite index normal subgroups of G is trivial. Equivalently, G is residually finite if given any nontrivial element $g \in G$, there exists a homomorphism from G to some finite group such that g is not in the kernel. A group is **hopfian** if every homomorphism from the group onto itself is an automorphism. A well-known theorem of Malcev states that every finitely generated, residually finite group is hopfian. The proof of the theorem (see page 197 of [LySc]) requires that the group be finitely generated only so that one knows that the number of subgroups of an arbitrary finite index r is finite. The following corollary is immediate.

COROLLARY 4.2: *Assume that* (G, χ) *is a group system. If* G *is residually finite and* $[\chi] \in \Sigma$, *then* K_χ *is hopfian.*

Any knot group G together with an abelianization homomorphism $\chi\colon G \to \mathbf{Z}$ comprises a group system. In this case, the kernel K_χ is simply the commutator subgroup G'.

QUESTION 4.3 ([GoWh]): *If* G *is the group of a knot in* S^3, *is the commutator subgroup* G' *hopfian?*

Question 4.3 provided the original motivation for our paper. In view of the fact that knot groups are residually finite [Th], W. Whitten and the first author had hoped to provide an affirmative answer to Question 4.3 by showing that the commutator subgroup of any knot group has only finitely many subgroups with an arbitrary finite index. The techiques of this paper, however, show that the commutator subgroup of the group of the knot 5_2 (Figure 6) has uncountably many subgroups of index r whenever $r > 3$. In order to see this, one applies the Reidemeister–Schreier method to a Wirtinger presentation

$$\begin{aligned}\langle x_1, x_2, x_3, x_4, x_5 \mid x_1 = x_2x_5x_2^{-1}, x_3 = x_5x_2x_5^{-1}, x_4 = x_1x_3x_1^{-1}, x_2 = x_4x_1x_4^{-1}\rangle \\ \cong \langle x_2, x_5 \mid x_2x_5x_2^{-1}x_5x_2x_5^{-1}x_2x_5^{-1}x_2^{-1}x_5x_2^{-1}x_5^{-1}x_2x_5^{-1}\rangle \\ \cong \langle x, a \mid xa^2x^{-1}x^2a^{-2}x^{-2}xax^{-1}a^{-2}\rangle,\end{aligned}$$

where $x = x_2$ and $a = x^{-1}x_5$, obtaining the following presentation for the commutator subgroup G',

$$\langle a_i \mid a_{i+1}^2 a_i^{-2} a_{i+1} a_{i+2}^{-2}\rangle.$$

The directed graph $\hat{\Gamma}$ that describes the representation shift (Φ_4, σ_x) is large, but we need only concern ourselves with the detail in Figure 7 that shows two cycles. Any bi-infinite path in $\hat{\Gamma}$ that travels at least once around each cycle corresponds to a transitive representation of G' in S_4. By Proposition 1.1 we conclude that G' contains uncountably many subgroups having index 4. Moreover, since each permutation in the inner cycle fixes 4, we can adjoin a third cycle, identical to the second but with '5' replacing '4', and thereby obtain a detail of the directed graph describing $(\Phi_5,\ \sigma_x)$ (see Figure 8.) Again, any bi-infinite path that travels at least once around each cycle corresponds to a transitive representation of G', this time in S_5. This process can be repeated in order to obtain uncountably many subgroups of G' having index r, for any $r \geq 4$. Interestingly, G' has only finitely many subgroups of index $r < 4$ (cf. Question 7.1).

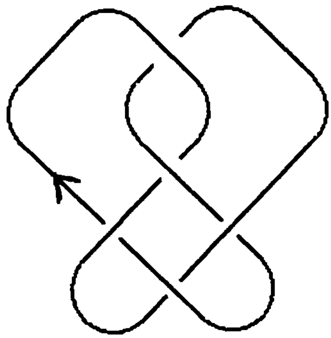

Figure 6. The knot 5_2.

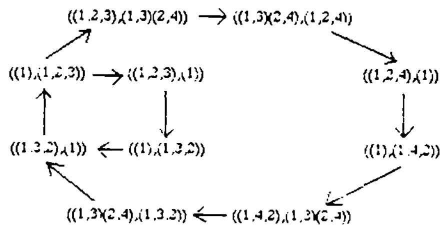

Figure 7.

Proof of Theorem 4.1: Assume that G is a finitely presented group and $\chi\colon G \to \mathbf{Z}$ is an epimorphism with kernel K_χ. Let X be the shift of finite type described in the proof of Theorem 3.1. If $[\chi] \in \Sigma$, then the associated subgroup $H^*_{0,1}$ coincides with the base H^*_0. Equivalently, for each i, $1 \leq i \leq n$, there is a word w_i in the generators $a_{i,j} \in A_{0,1}$ such that $a_{i,M_i} = w_i$ is a consequence of the relators $\bigcup_{t\in\mathbf{Z}} R_t$ of K_χ. Choose a nonnegative integer q large enough so that $a_{i,M_i} = w_i$ in $H_{[-q,q]}$, $1 \leq i \leq n$. Suppose that $\rho = (\rho_j)$ is any element of X. For any j the $(2q+1)$-block $\rho_{j-q}, \ldots, \rho_{j+q}$ corresponds to a representation of $H_{[-q,q]}$ in S_r. Since $a_{i,M_i} = w_i$, $1 \leq i \leq n$ in $H_{[-q,q]}$, the values $\rho_j(a_{1,M_1}), \ldots, \rho_j(a_{n,M_n})$ are uniquely determined by the values $\rho_j(a_{i,j}), a_{i,j} \in A_{0,1}$. In terms of the pruned graph $\hat{\Gamma}$ representing X (see discussion following Example 3.3) this means that any vertex has exactly one directed edge leaving it. Hence the shift X is finite. ∎

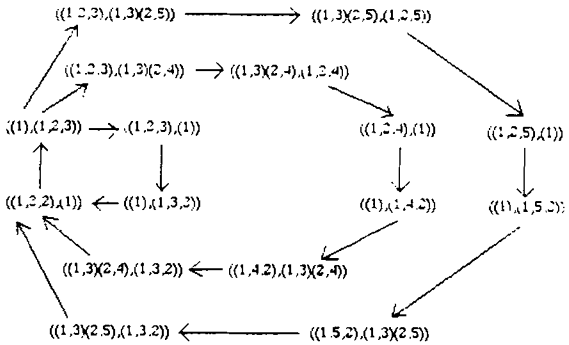

Figure 8.

5. The Baumslag–Solitar groups

In 1962 Baumslag and Solitar proved that the group

$$G(m,n) = \langle x,\ a \mid xa^m x^{-1} = a^n \rangle$$

is nonhopfian whenever $m, n \geq 2$ are coprime [BaSo]. For any integers m and n, we define χ: $G(m,n) \to \mathbf{Z}$ to be the homomorphism such that $\chi(x) = 1$ and $\chi(a) = 0$. The kernel $K(m,n)$ of χ has presentation $\langle a_i \mid a_i^m = a_{i+1}^n \rangle$. We apply the techniques of Section 3 in order to prove the following.

PROPOSITION 5.1:

(i) *Assume that $(m,n) = 1$. Then $K(m,n)$ has a subgroup of index r if and only $(r,m) = (r,n) = 1$. In this case, there is exactly one such subgroup.*

(ii) *Assume that $(m,n) \neq 1$. If r is greater than or equal to the smallest prime divisor of both m and n, then $K(m,n)$ has uncountably many subgroups of index r.*

Proof: From the relator $a_i^m = a_{i+1}^n$ we have $a_i^{m^2} = a_{i+1}^{mn} = a_{i+2}^{n^2}$, and by induction

$$a_i^{m^k} = a_{i+k}^{n^k} \tag{5.2}$$

for all $k \geq 1$. Let ρ be a transitive representation of $K(m,n)$ in S_r, and denote the permutation $\rho(a_i)$ by ρ_i. Since S_r is finite, $\rho_s = \rho_{s+t}$ for some s and some

$t \geq 1$, and hence $\rho_s^{m^t} = \rho_s^{n^t}$ by (5.2). Let l be the order of ρ_s in S_r. Then l divides $m^t - n^t$. Since $(m,n) = 1$, we must have $(l,m) = (l,n) = 1$.

We claim that all of the permutations ρ_i have order l. From this claim it follows that we can solve the equation $\rho_i^m = \rho_{i+1}^n$ in order to obtain $\rho_{i+1} = \rho_i^{mq}$, where q is the inverse of n modulo l; or, going backwards, $\rho_i = \rho_{i+1}^{nv}$, where v is the inverse of m modulo l.

Consequently, all of the ρ_i are uniquely determined as powers of the single permutation ρ_0. Since the representation ρ is transitive, ρ_0 must act transitively on $\{1, \ldots, r\}$, so $l = r$ and, up to inner automorphism of S_r, we can assume that $\rho_0 = (1, 2, \ldots, r)$. From this and the comment concluding Section 1 follows the necessity of the condition $(r,m) = (r,n) = 1$ as well as the uniqueness of the index r subgroup when the condition is met.

We now prove the claim. It suffices to show that if some ρ_i has order l, with $(l,m) = (l,n) = 1$, then ρ_{i+1} and ρ_{i-1} also have order l. We appeal to the following number-theortic lemma, the proof of which is left to the reader.

LEMMA 5.3: *Suppose that g is any finite-order element of a group. If the order $o(g^a)$ is equal to b, and if $(a,b) = d$, then $o(g) = a_1 db$ for some a_1 dividing a/d.*

Continuing the proof of Proposition 5.1, assume that ρ_i has order l. Since $(l,m) = 1$, we have $l = o(\rho_i^m) = o(\rho_{i+1}^n)$. Using Lemma 5.3, $o(\rho_{i+1}) = n_1 l$, where n_1 divides n. Then $n_1 l = o(\rho_{i+1}^m) = o(\rho_{i+2}^n)$. Again using the lemma, $o(\rho_{i+2}) = n_2 n_1^2 l$, where n_2 divides n/n_1. Continuing in this manner, we find that the order of ρ_{i+k} is divisible by n_1^k. Since the order is bounded by r, we must have $n_1 = 1$ and $o(\rho_{i+1}) = l$. The same argument, exchanging the roles of m and n, shows that $o(\rho_{i-1}) = l$. This completes the proof of (i).

We now prove (ii). Let $p > 1$ divide m and n, and suppose that $r \geq p$. Let S be the set of elements of S_r that are products of disjoint p-cycles. We include the "empty product" (1), so the cardinality of S is greater than 1 even if $r = p = 2$. Then (Φ_r, σ_x) contains the full shift on the elements of S. It is easy to see that uncountably many elements of this shift correspond to transitive representations of $K(m,n)$ in S_r. Using Proposition 1.1 we find that $K(m,n)$ contains uncountably many subgroups of index r. ∎

6. An entropy invariant for (G, χ)

In Section 4 we saw that if (G, χ, x) is an augmented group system, then some conclusions about the associated representation shifts $(\Phi_r,\ \sigma_x)$ can be deduced from the pair (G,χ) alone. Is the conjugacy class of the shift (Φ_r,σ_x), in fact, independent of the choice of distinguished element x? Example 6.1 shows that the answer is no.

Example 6.1: Consider the augmented group system $(G,\ \chi,\ x)$, where G is the free group on x,a, and $\chi\colon G\to\mathbf{Z}$ is the epimorphism determined by $x\mapsto 1$, $a\mapsto 0$. The kernel K_χ is free on generators a_i, $i\in\mathbf{Z}$, where a_i denotes $x^{-i}ax^i$. Clearly the associated representation shift $(\Phi_3,\ \sigma_x)$ is conjugate to the full 6-shift. This shift has exactly 6 fixed points, corresponding to the representations $\rho\colon a_i\mapsto\pi$, for all $i\in\mathbf{Z}$, where π is any permutation in S_3.

Now consider the augmented group system $(G,\ \chi,\ y)$, where $y=xax^{-1}a^{-1}x$. In order to study the associated representation shift (Φ_3,σ_y), we first apply a sequence of Tietze transformations (see [LySc]) to the presentation $\langle x,a|\ \rangle$ for G:

$$\langle x,a|\ \rangle\cong\langle x,a,b,y|\ b=ax^{-1}a^{-1}x,y=xb\rangle\cong\langle a,b,y|\ b=aby^{-1}a^{-1}yb^{-1}\rangle.$$

Next we apply the Reidemeister–Schreier method to the last presentation. We obtain a new presentation $\langle a_i,b_i|\ b_i=a_ib_ia_{i+1}^{-1}b_i^{-1}\rangle$ for K_χ, where a_i now denotes $y^{-i}ay^i$, and likewise b_i denotes $y^{-i}by^i$. It is easy to check that (Φ_3,σ_y) has 6 fixed points corresponding to the representations $\rho\colon a_i\mapsto\pi, b_i\mapsto(1)$, where $\pi\in S_3$. However, $(\Phi_3,\ \sigma_y)$ has 6 more fixed points:

$$\begin{aligned}
a_i&\mapsto(1,2),\quad b_i\mapsto(1,2,3),\\
a_i&\mapsto(2,3),\quad b_i\mapsto(1,2,3),\\
a_i&\mapsto(1,3),\quad b_i\mapsto(1,2,3),\\
a_i&\mapsto(1,2),\quad b_i\mapsto(1,3,2),\\
a_i&\mapsto(2,3),\quad b_i\mapsto(1,3,2),\\
a_i&\mapsto(1,3),\quad b_i\mapsto(1,3,2).
\end{aligned}$$

Since conjugate shifts have the same number of fixed points, (Φ_3,σ_x) and (Φ_3,σ_y) are not conjugate.

Definition 6.2: Shifts of finite type X and X' are **finitely equivalent** if there exists a shift of finite type that maps onto each by mappings that are finite-to-one.

Finite equivalence is a weaker form of equivalence than conjugacy [LiMa]. Nevertheless, it is a useful notion. The **entropy** of a shift can be defined as

$\limsup 1/N \log |\mathcal{B}_N|$, where $|\mathcal{B}_N|$ is the number of allowable N-blocks of the shift. When the shift is described by a directed graph Γ, its entropy is $\log \lambda$, where λ is the Perron eigenvalue of the adjacency matrix of Γ (see Chapter 4 of [LiMa] for details). Conjugate shifts have the same entropy. In fact, finitely equivalent shifts also have the same entropy.

THEOREM 6.3: *Assume that (G, χ, x) and (G, χ, y) are augmented group systems that differ only by the choice of distinguished elements x and y. Then for each $r > 0$, the associated shifts (Φ_r, σ_x) and (Φ_r, σ_y) are finitely equivalent.*

Proof: Recall that Φ_r is the set of representations $\rho\colon K_\chi \to S_r$. The mappings σ_x and σ_y are defined by $\sigma_x\rho(g) = \rho(x^{-1}gx)$ and $\sigma_y\rho(g) = \rho(y^{-1}gy)$, for all $g \in K_\chi$. We can write $y = xb$ for some element $b \in K_\chi$. The proof of the following lemma is straightforward, and we leave it to the reader.

LEMMA 6.4: *For $n > 0$,*

$$\sigma_y^n\rho(g) = [\rho(b_{n-1}\cdots b_1b_0)]^{-1}\sigma_x^n\rho(g)\rho(b_{n-1}\cdots b_1b_0),$$
$$\sigma_y^{-n}\rho(g) = \rho(b_{n-1}\cdots b_1b_0)\sigma_x^n\rho(g)\rho[(b_{n-1}\cdots b_1b_0)]^{-1},$$

where $b_n = x^{-n}bx^n$.

Define Θ_r to be the product space $\Phi_r \times S_r$, where S_r has the discrete topology, and define $T\colon \Theta_r \to \Theta_r$ by $T(\rho, \pi) = (\sigma_x\rho, \rho(b)\pi)$. One easily checks that T is continuous. (The form of the mapping is a "skew product" with base (Φ_r, σ_x).) Note that $T^n(\rho,\pi) = (\sigma_x^n\rho, \rho(b_{n-1}\cdots b_0)\pi)$ and $T^{-n}(\rho,\pi) = (\sigma_x^{-n}\rho, [\rho(b_{n-1}\cdots b_0)]^{-1}\pi)$, if $n > 0$. It follows that the first-coordinate projection $p_1\colon \Theta_r \to \Phi_r$ induces an $r!$-to-1 mapping from (Θ_r, T) onto (Φ_r, σ_x). Also, $f\colon \Theta_r \to \Phi_r$ defined by $f(\rho, \pi) = \pi^{-1}\rho\pi$ induces a mapping from (Θ_r, T) onto (Φ_r, σ_y). The mapping f is also $r!$-to-1 since $f^{-1}(\rho) = \{(\pi\rho\pi^{-1}, \pi) \mid \pi \in S_r\}$.

In order to complete the proof of Theorem 6.3 it suffices to show that the dynamical system (Θ_r, T) is conjugate to a shift of finite type. As in the proof of Theorem 3.1, obtain a presentation $\langle a_{i,j} \mid R_j\rangle$ for K_χ corresponding to a presentation $\langle x, a_i \mid R_0\rangle$ for G. Map Θ_r to a shift with alphabet $\{(\rho_0, \pi) \mid \rho_0$ is a function from A_0 to S_r, $\pi \in S_r\}$ by sending (ρ, π) to (ρ_i, π_i), where $\rho_i = \sigma_x^i\rho|_{A_0}$, $\pi_0 = \pi$, and (inductively) $\pi_{i+1} = \rho(b_i)\pi_i$. If $b = b_0 = w(a_{-l}, \ldots, a_l)$ is a word in the a_i, then $b_i = w(a_{i-l}, \ldots, a_{i+l})$ and $\rho(b_i) = w(\rho_{i-l}, \ldots, \rho_{i+l})$. Thus the condition $\pi_{i+1} = \rho(b_i)\pi$ is a finite type condition (i.e., a condition that one can verify by examining blocks of a fixed length), and the image of Θ_r is a shift of finite type. It is easy to check that this mapping is a conjugacy. ∎

Remark 6.5: The above proof shows more than the statement of Theorem 6.3. There exists a shift of finite type that maps constant-to-one onto each of (Φ_r, σ_x) and (Φ_r, σ_y).

Definition 6.6: A **group system** is a pair (G, χ) consisting of a finitely presented group G and an epimorphism $\chi: G \to \mathbf{Z}$. Two group systems (G, χ) and (G', χ') are **isomorphic** if there exists a group isomorphism $h: G \to G'$ such that $\chi = \chi' \circ h$.

COROLLARY 6.7: *Let (G, χ, x) be an augmented group system, and let r be a positive integer. The entropy $h(\Phi_r)$ of the associated representation shift (Φ_r, σ_x) is an invariant of the group system (G, χ); i.e., the entropy depends only on the isomorphism class of the group system.*

7. Questions and further directions

PROBLEM 7.1: *Characterize those augmented group systems (G, χ, x) such that (Φ_r, σ_x) is finite for all r.*

Corollary 6.7 can be used to define a sequence of "entropy invariants" for n-knots. An n-knot is a smoothly embedded n-sphere K in the $(n+2)$-sphere S^{n+2}. Here n-knots are assumed to be oriented. Let $N(K)$ be a neighborhood of K that is diffeomorphic to $S^n \times D^2$. The closure $X(K)$ of $S^{n+2} - N(K)$ is called the **exterior** of K. Two n-knots are **equivalent** if there is a diffeomorphism of S^{n+2} to itself that sends one n-knot to the other (preserving orientations.) Equivalent n-knots are regarded as the same. (See [Ro] for additional background information.) An n-knot **invariant** is a quantity that is defined for an n-knot and depends on the n-knot only up to equivalence. As in [Sil] every n-knot K determines an augmented group system: Let $G = \pi_1(X(K), *)$, where the basepoint $*$ lies in the boundary $\partial X(K)$, and let x be the element of G represented by a simple closed curve $m \subset \partial X(K)$ with its orientation induced by K (the curve m is called a **meridian** of K). By the uniqueness up to isotopy of tubular neighborhoods, the element x is well defined by K. Letting $\chi: G \to \mathbf{Z}$ be the abelianization homomorphism that sends x to 1, we obtain an augmented group system (G, χ, x). Associated now to K is a sequence $\{\Phi_r(K)\}_{r=1}^{\infty}$ of shifts and a corresponding (nondecreasing) sequence of entropies $\{h_r(K)\}_{r=1}^{\infty}$.

QUESTION 7.2: *How do the invariants $h_r(K)$ relate to previously defined n-knot invariants (e.g., Alexander module, knot entropy* [Si2]*)?*

The invariants $h_r(K)$ are computable from any knot diagram for K when K is a 1-knot. We will investigate this in a future paper.

Reversing the orientation of K produces a new (oriented) n-knot $\mathcal{R}K$. The augmented group system of $\mathcal{R}K$ is $(G, -\chi, x^{-1})$. It is not difficult to see that for any $r \geq 1$ there is a bijection from $\Phi_r(K)$ to $\Phi_r(\mathcal{R}K)$, sending every bi-infinite word (ρ_j) to its "reverse" (ρ_{-j}). From this it follows that the entropies $h_r(K)$ and $h_r(\mathcal{R}K)$ are equal, and hence $h_r(K)$ is an invariant of the *unoriented* n-knot K. However, the shifts $\Phi_r(K)$ and $\Phi_r(\mathcal{R}K)$ need not be conjugate.

QUESTION 7.3: *How often do the shifts $\{\Phi_r(K)\}_{r=1}^{\infty}$ distinguish an n-knot K from its inverse $\mathcal{R}K$?*

Growth rates of finitely generated group automorphisms were used in [Si3] in order to distinguish n-knots K from their inverses. While the techniques used there are very effective, they apply only in the case that the commutator subgroup of $\pi_1(X(K))$ is finitely generated, and they can be effective only when n is greater than 1.

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